REFLECTION OF MAGNETO-ACOUSTIC WAVES AT THE BOUNDARY BETWEEN TWO MEDIA WITH FINITE ELECTRIC CONDUCTIVITY

(OTRAZHENIE MAGNITOZVUKOVYKH VOLN NA GRANITSE RAZDELA DVUKH SRED S KONECHNOI ELEKTROPROVODNOST'IU)

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L.Ia. KOSACHEVSKII
(Donetsk)

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In the paper [1], the problem of the reflection of magneto-acoustic waves from a plane boundary separating a fluid and an elastic medium with infinite electrical conductivity was solved. Below, the analogous problem for media with finite electrical conductivity is solved. The amplitude coefficients for reflection and refraction are calculated for the case of a magnetic field. To the same approximation, the question of the change of the form of reflected pulses, due to dependence of the reflection coefficients on frequency, is investigated.

1. Basic equations. We shall assume that the fluid and elastic media are homogeneous and are in a homogeneous steady magnetic field **H** . We neglect the viscosity and heat conductivity of the media.

The propagation of plane waves in the fluid is described by Equations [2]

$$\mathbf{k}\times\mathbf{E}=\frac{\omega}{c}\,\mathbf{h},\qquad \mathbf{E}=\frac{ic}{4\pi\sigma_0}\,\mathbf{k}\times\mathbf{h}-\frac{1}{c}\,\mathbf{v}\times\mathbf{H},\qquad \mathbf{k}\cdot\mathbf{h}=0$$

$$\omega\mathbf{v}=\frac{p}{\rho_0}\,\mathbf{k}+\frac{1}{4\pi\rho_0}\,\mathbf{H}\times(\mathbf{k}\times\mathbf{h}),\quad \omega p=\rho_0 a_0^2\ \mathbf{k}\cdot\mathbf{v} \tag{1.1}$$

Here, **k** is the wave vector, ω the frequency, **E** the intensity of the induced electric field, **h** the small change in magnetic field intensity in the wave, **v** the velocity of the fluid, p the hydrodynamic pressure, ρ_0 and σ_0 the density and conductivity, respectively, of the fluid, a_0 the ordinary speed of sound in the fluid, c the velocity of light. The magnetic permeability of both media is taken to be unity. We take the xy plane to be the boundary between the two media. The vectors **H** and **k** will be considered to be in the xz plane.

For waves in which the vectors **v** , **h** lie in the xz plane, Equations (1.1) lead to the dispersion equation

$$u^2-(1+\psi_0)\,u+\psi_0\cos^2\alpha+i\omega\eta_0\,(u-1)=0 \tag{1.2}$$

$$u=(\omega/ka_0)^2,\qquad \psi_0=H^2/4\pi\rho_0a_0^2,\qquad \eta_0=c^2/4\pi\sigma_0a_0^2$$

Here, u and ψ_0 represent the squares of the phase velocity and the magnetic field intensity in dimensionless form, α is the angle between the vectors **k** and **H** .

The two roots, u_1 and u_2, of Equation (1.2) correspond to fast and slow magneto-acoustic waves for small $\omega\eta_0$. We shall restrict ourselves to the case of weak magnetic field, $\psi_0 \ll 1$, with accuracy up to small quantities of the first order, we have

$$u_1 = 1 + \psi_0 \sin^2 \alpha, \qquad u_2 = \psi_0 \cos^2 \alpha - i\omega\, \eta_0 \tag{1.3}$$

Thus, to the approximation made, the first wave propagates without attenuation with velocity somewhat higher than the ordinary sound speed in the fluid. The second wave will be an attenuated one.

From Equations (1.1), we obtain

$$h_{\nu x} = A_\nu v_{\nu z}, \qquad E_{\nu y} = B_\nu v_{\nu z}, \qquad -p = Z_\nu v_{\nu z} \quad (\nu = 1,2) \tag{1.4}$$

$$A_\nu = H\,(u_\nu - 1)\, \sqrt{u_\nu} k_{\nu z} \,/\, a_0 \psi_0 \beta_\nu, \quad B_\nu = -H\,(u_\nu - 1)\, u_\nu k_\nu \,/\, c\psi_0 \beta_\nu$$

$$Z_\nu = -\rho_0 a_0 \sqrt{u_\nu} k_\nu \beta_\nu^{-1} \sin \alpha_\nu, \qquad \beta_\nu = k_\nu u_\nu \cos \varphi - k_{\nu x} \cos \alpha_\nu$$

where φ is the angle of inclination of the magnetic field $\mathbf{H}$ to the x-axis, Z_ν and B_ν / A_ν are the acoustic and electromagnetic impedances.

Waves with the vectors $\mathbf{v}$, $\mathbf{h}$ perpendicular to the xz plane (Alfven waves) propagate independently of the magneto-acoustic ones and, in our approximation, have a velocity which coincides with the velocity of the slow magneto-acoustic wave.

The propagation of waves in an unbounded elastic conducting medium was investigated in [3 and 4]. The equations for plane waves in this medium may be written in the form

$$-\omega \mathbf{h} = \mathbf{k} \times (\mathbf{v} \times \mathbf{H}) + ia^2\eta k^2 \mathbf{h}$$

$$-(\omega^2/a^2)\, v = \mathbf{k}\,(\mathbf{k}\cdot\mathbf{v}) + \xi \mathbf{k} \times (\mathbf{k} \times \mathbf{v}) - (\omega\psi/H^2)\, \mathbf{H} \times (\mathbf{k} \times \mathbf{h}) \tag{1.5}$$

$$P_{zz} = -\omega^{-1} [\lambda\, (\mathbf{kv}) + 2\mu k_z v_z], \quad P_{xz} = -\mu\omega^{-1} (k_z v_x + k_x v_z)$$

$$a^2 = (\lambda + 2\mu)\,/\,\rho, \quad b^2 = \mu\,/\,\rho, \quad \xi = b^2\,/\,a^2, \quad \eta = c^2\,/\,4\pi\sigma a^2, \quad \psi = H^2\,/\,4\pi\rho a^2$$

Here, a and b are the velocities of purely elastic longitudinal and transverse waves, ρ, λ and μ are the density and the Lamé constants of the elastic medium, σ is the conductivity, P_{zz} and P_{xz} are components of the stress tensor of the elastic medium. For waves which are polarized in the xz plane, we have the dispersion equation

$$u^2 - (1 + \xi + \psi)\, u + \xi + \psi\, (\cos^2 \alpha + \xi \sin^2 \alpha) + i\omega\eta u^{-1}\, (u - 1)\, (u - \xi) = 0, \quad u = (\omega\,/\,ka)^2 \tag{1.6}$$

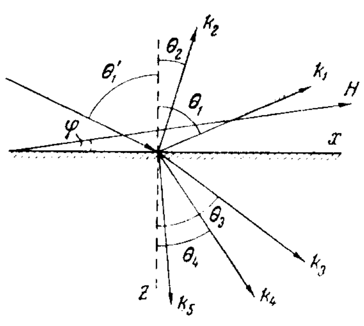

Fig. 1

For small $\omega\eta$ and ψ, the roots of this equation will be

$$u_3 = 1 + \psi \sin^2 \alpha, \qquad u_4 = \xi + \psi \cos^2 \alpha$$

$$u_5 = -i\omega\eta \tag{1.7}$$

u_3 and u_4 correspond to fast and slow magneto-elastic waves, which in the limit $\psi = 0$ become purely elastic longitudinal and transverse waves. The attenuation of these waves may be neglected. The root u_5 corresponds to a strongly attenuated wave, which vanishes for infinitely large conductivity of the medium. According to (1.5), for waves in an elastic medium, we have

$$h_{\nu x} = A_\nu v_{\nu z}, \quad E_{\nu y} = B_\nu v_{\nu z}, \quad P_{\nu zz} = Z_\nu v_{\nu z}, \quad P_{\nu xz} = X_\nu v_{\nu z}$$

$$A_\nu = (H\,/\,a\psi\beta_\nu)\, u_\nu^{-1/2} (u_\nu - 1)\, (u_\nu - \xi)\, k_{\nu z}, \quad B_\nu = -\,(H\,/\,c\psi\beta_\nu)\, (u_\nu - 1)\, (u_\nu - \xi) \tag{1.8}$$

$$Z_\nu = -\,(\rho a\,/\,\beta_\nu)\, u_\nu^{-1/2} [(u_\nu - \xi)\, (k_\nu \sin \alpha_\nu - 2\xi k_{\nu x} \sin \varphi) - 2\xi\, (1 - \xi)\, k_{\nu x} k_{\nu z} k_\nu^{-1} \cos \alpha_\nu]$$

$$X_\nu = -(\rho a \xi / \beta_\nu)\, u_\nu^{-1/2} [(u_\nu - 1)(k_\nu \cos\alpha_\nu + 2k_{\nu z} \sin\varphi) + 2(1-\xi) k_{\nu x} k_{\nu z} k_\nu^{-1} \sin\alpha_\nu] \quad (1.8) \text{ cont.}$$

$$\beta_\nu = k_\nu (u_\nu - \xi) \cos\varphi - (1-\xi) k_{\nu x} \cos\alpha_\nu \qquad (\nu = 3, 4, 5)$$

For waves which are polarized perpendicular to the xz plane, the dispersion equation has the form

$$u^2 - u(\xi + \psi\cos^2\alpha) + i\omega\eta(u - \xi) = 0$$

Its roots

$$u_6 = \xi + \psi\cos^2\alpha, \qquad u_7 = -i\omega\eta$$

correspond to a slightly modified transverse wave and a strongly attenuated electromagnetic wave which vanishes completely for infinite conductivity.

2. Reflection of magneto-acoustic waves. At the boundary separating two media, continuity is required of the normal component of velocity, the tangential components of the magnetic and electric fields and the normal pressure. Tangential stresses should be absent. For waves polarized in the xz plane, these conditions give

$$[v_z] = [h_x] = [E_y] = 0, \qquad P_{zz} = -p,\ P_{xz} = 0 \qquad (2.1)$$

where $[v]$ is the jump in the quantity v at the boundary.

Let a fast magneto-acoustic wave from the fluid impinge on the boundary (Fig.1). Quantities relating to the incident disturbance will be denoted by a prime. To satisfy the five boundary conditions (2.1), it is necessary to assume that on the boundary the incident wave produces a system of five waves – two magneto-acoustic ones in the fluid and three in the elastic medium. The angles of incidence, reflection and refraction are connected by the relation (Snell's law)

$$\frac{\sin\theta_1'}{\sqrt{u_1'}} = \frac{\sin\theta_1}{\sqrt{u_1}} = \frac{\sin\theta_2}{\sqrt{u_2}} = \frac{a_0 \sin\theta_3}{a\sqrt{u_3}} = \frac{a_0 \sin\theta_4}{a\sqrt{u_4}} = \frac{a_0 \sin\theta_5}{a\sqrt{u_5}}$$

According to (1.3) and (1.7), with accuracy up to the principal terms, we have

$$\sin\theta_1' = \sin\theta_1 = \frac{\sin\theta_2}{\sqrt{\psi_0 \sin^2\varphi - i\omega\eta_0}} = \frac{a_0}{a}\sin\theta_3 = \frac{a_0}{b}\sin\theta_4 = \frac{a_0 \sin\theta_5}{a\sqrt{-i\omega\eta}} \qquad (2.2)$$

Taking the amplitude v_{1z}' to be unity and taking into account (1.4) and (1.8), we obtain from (2.1) a system of linear equations for the amplitude coefficients of reflection and refraction W_ν for $v_{\nu z}$

$$\sum_{\nu=1}^{5} W_\nu = 1, \quad \sum_{\nu=1}^{5} A_\nu W_\nu = A_1', \quad \sum_{\nu=1}^{5} B_\nu W_\nu = B_1', \quad \sum_{\nu=1}^{5} Z_\nu W_\nu = Z_1' \quad \sum_{\nu=3}^{5} X_\nu W_\nu = 0$$

Up to the principal terms, the solution of this system has the form

$$W_1 = W_1^\circ - \frac{\psi_0 a_0 Y^2 \cos\theta_1}{2\Omega}, \qquad W_2 = \frac{\psi_0}{\Omega} a_0 Y \sin\varphi \sin\theta_1 \qquad (2.3)$$

$$W_3 = (1 - W_1^\circ)\cos 2\theta_4 + \frac{\psi a Y \sin\varphi \sin\theta_3}{\Omega(Z_n + Z_1)}\left[Z_1 + Z_3 \frac{\sin(2\theta_4 - \theta_1)}{\cos\theta_1}\cot\theta_3\right]$$

$$W_4 = 2(1 - W_1^\circ)\sin^2\theta_4 - \frac{\psi a Y \sin\varphi \sin\theta_3}{\Omega(Z_n + Z_1)}(Z_1 + Z_3\cos 2\theta_4 + Z_4 \sin 2\theta_4 \tan\theta_1)$$

$$W_5 = -\frac{\psi}{\Omega}\sqrt{-i\omega\eta}\, aY, \qquad \Omega = a\sqrt{-i\omega\eta} + a_0\sqrt{\psi_0 \sin^2\varphi - i\omega\eta_0}$$

$$Y = \frac{2 \sin \varphi}{Z_n + Z_1} \left[Z_n \tan \theta_1 + Z_1 \frac{\sin (2\theta_4 - \theta_3)}{\cos \theta_3} \right] \qquad \text{(2.3) cont.}$$

$$W_1^\circ = \frac{Z_n - Z_1}{Z_n + Z_1}, \qquad Z_n = Z_3 \cos^2 2\theta_4 + Z_4 \sin^2 2\theta_4$$

$$Z_1 = \frac{\rho_0 a_0}{\cos \theta_1}, \qquad Z_3 = \frac{\rho a}{\cos \theta_3}, \qquad Z_4 = \frac{\rho b}{\cos \theta_4}$$

The quantities W_1° and Z_n represent, respectively, the coefficient of reflection and the total acoustic impedance of the boundary in the absence of the magnetic field.

Equations (2.3) show that a weak magnetic field changes the ordinary acoustic coefficients of reflection and refraction by a small quantity of order $\psi_0 a_0/\Omega$ and $\psi a/\Omega$, respectively. The coefficient W_5 is a small quantity of much higher order, $\psi a \Omega^{-1} (\omega\eta)^{1/2}$, and, therefore, in the first approximation, it may be assumed that the incident wave excites only two magneto-elastic waves in the elastic medium. For normal incidence of the wave on the boundary, the magnetic field has no effect on the coefficients W_ν ($Y = 0$). For inclined incidence, only a magnetic field parallel to the boundary produces no effect.

3. Reflection of magneto-acoustic pulses. Now let us assume that a pulse of the form [5]

$$p(\zeta) = \frac{\varepsilon}{\varepsilon^2 + \zeta^2}, \quad \zeta = \frac{x \sin \theta_1' + z \cos \theta_1'}{a_0 \sqrt{u_1'}} - t \approx \frac{x \sin \theta_1 + z \cos \theta_1}{a_0} - t$$

having a plane front, impinges from the fluid onto the separation boundary. Here, p is the pressure, ε is a parameter characterizing the width of the pulse. A Fourier integral resolution of the incident and the first reflected pulses has the form

$$p(\zeta) = \mathrm{Re} \int_0^\infty e^{-(\varepsilon - i\zeta)\omega} d\omega, \qquad p_1(\zeta_1) = \mathrm{Re} \int_0^\infty W_1 e^{-(\varepsilon - i\zeta_1)\omega} d\omega$$

$$\zeta_1 \approx \frac{x \sin \theta_1 - z \cos \theta_1}{a_0} - t \qquad (3.1)$$

For infinite conductivity of both media and angles of incidence less than the limiting angles for total internal reflections the coefficient of reflection W_1 of a plane harmonic wave is real and does not depend on frequency. Therefore, the form of the reflected pulse will be identical to that of the incident one.

Let us investigate the change of form of the reflected pulse for finite electrical conductivity in the elastic medium and for given angles of incidence. We take the conductivity of the fluid medium to be infinite ($\eta_0 = 0$).

According to (3.1) and (2.3),

$$p_1(\zeta_1) = W_1^\circ p(\zeta_1) - {}^1/_2\, \psi_0 a_0 Y^2 \cos \theta_1 \,\mathrm{Re}\, J_1 \qquad (3.2)$$

$$J_1 = \int_0^\infty \frac{\exp(-\mu\omega)\, d\omega}{a\sqrt{-i\omega\eta} + a_0 \sqrt{\psi_0} \sin \varphi} = \frac{1}{a\sqrt{-i\eta}} \left[\int_0^\infty \frac{\sqrt{\omega} \exp(-\mu\omega)\, d\omega}{\omega - i\beta} - \right.$$

$$\left. - \sqrt{i\beta} \int_0^\infty \frac{\exp(-\mu\omega)\, d\omega}{\omega - i\beta} \right], \qquad \mu = \varepsilon - i\zeta_1, \qquad \beta = \frac{a_0^2 \psi_0 \sin^2 \varphi}{a_0^2 \eta}$$

Making the substitution $\omega = x^2$ in the first integral in the square brackets, we obtain [6]

$$J_1 a \sqrt{\eta} = \sqrt{\pi i / \mu} - \sqrt{\beta} \left[2 \sqrt{\pi} \operatorname{Erfc}\left(\sqrt{-i\beta\mu}\right) - i \operatorname{Ei}(i\beta\mu)\right] \exp(-i\beta\mu)$$

$$\operatorname{Erfc}(x) = \tfrac{1}{2}\pi \left[1 - \Phi(x)\right] \qquad (3.3)$$

Here, $\Phi(x)$ is the probability integral, $\mathrm{Ei}(x)$ is the integral exponential function. If $\beta|\mu| \gg 1$, use can be made of the asymptotic formula

$$\operatorname{Erfc}(\sqrt{x}) \approx \frac{1}{2\sqrt{x}}\left(1 - \frac{1}{2x}\right) e^{-x}, \qquad \operatorname{Ei}(-x) \approx -\frac{1}{x}\left(1 - \frac{1}{x}\right) e^{-x}$$

and the expression for the reflected pulse takes the form

$$p_1(\zeta_1) = V_1 p(\zeta_1) + \frac{a \sqrt{\eta}\, Y^2 \cos\theta_1}{a_0 \sin^2\varphi\, (\varepsilon^2 + \zeta_1^2)^{3/2}} \left\{ \frac{\sqrt{\pi}}{8} \left[(\varepsilon + \zeta_1) \sqrt{\sqrt{\varepsilon^2 + \zeta_1^2} + \varepsilon} + \right.\right.$$

$$\left.\left. + (\varepsilon - \zeta_1) \sqrt{\sqrt{\varepsilon^2 + \zeta_1^2} - \varepsilon}\right] - \frac{\varepsilon \zeta_1}{\sqrt{\beta (\varepsilon^2 + \zeta_1^2)}} \right\} \qquad (3.4)$$

$$V_1 = W_1^\circ - \frac{\sqrt{\psi_0}\, Y^2 \cos\theta_1}{2 \sin\varphi}$$

According to (2.3), the quantity V_1 represents the coefficient of reflection of a strong magneto-acoustic wave from the boundary between media with infinite conductivity. The first term of Expression (3.4) corresponds to a reflected pulse of the same form as the incident one. The remaining terms correspond to pulses of changing form.

Putting $\zeta_1 = 0$, we determine from (3.3) the pressure at the front of the reflected pulse

$$p_1(0) = W_1^\circ p(0) - \tfrac{1}{2} \sqrt{\psi_0}\, \beta Y^2 \csc\varphi \left\{ \sqrt{\tfrac{1}{2}\pi / \beta\varepsilon} - \pi \left[\cos\beta\varepsilon - (\cos\beta\varepsilon - \sin\beta\varepsilon)\right.\right.$$

$$\left.\left. C(\sqrt{\beta\varepsilon}) - (\cos\beta\varepsilon + \sin\beta\varepsilon) S(\sqrt{\beta\varepsilon})\right] + \operatorname{ci}(\beta\varepsilon) \sin\beta\varepsilon - \operatorname{si}(\beta\varepsilon)\cos\beta\varepsilon \right\} \qquad (3.5)$$

$$p(0) = 1 / \varepsilon$$

Here, $S(x)$ and $C(x)$ are Fresnel integrals, $\mathrm{si}(x)$ and $\mathrm{ci}(x)$ are the integral sine and cosine.

For every narrow pulses ($\varepsilon \to 0$), Equation (3.5) gives

$$p_1(0) = W_1^\circ p(0) - \frac{\psi_0 a_0 Y^2}{2a\sqrt{\eta}} \sqrt{\tfrac{1}{2}\pi p(0)} \qquad (3.6)$$

For $\beta\varepsilon \gg 1$ we have from (3.4),

$$p_1(0) = V_1 p(0) + \frac{a \sqrt{2\pi\eta}\, Y^2 \cos\theta_1}{8 a_0 \sin^2\varphi} p^{3/2}(0) \qquad (3.7)$$

The dependence of the coefficients W_2, W_3 and W_4 on frequency is the same as in W_1. The formulas for the second reflected pulse and the pulses transmitted into the elastic medium are analogous to (3.3) to (3.7)

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